SI sample mean properties

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① $\bar{X}$ and $S_n^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$ are independent.

思路：记 $\bar{X} \perp\!\!\!\perp S^2$ 转化为记 $\bar{X} \perp\!\!\!\perp x_i - \bar{x}$

转化为记 $\underbrace{M_{(\bar{x}_n, x_i-\bar{x}_n)}(s,t)}_{③} = \underbrace{M_{\bar{x}_n}(s)}_{①}\underbrace{M_{x_i-\bar{x}_n}(t)}_{②}$

先写RHS，因为简单.

$M_{\bar{x}_n}(t) \leftarrow \bar{X}_n$ sample mean $\sim N(u, \frac{\sigma^2}{n})$

Normal MGF: $e^{ut+\sigma^2t^2/2}$

① $M_{\bar{x}_n}(s) = e^{us + \frac{\sigma^2}{n}\cdot s^2/2}$

$x_i \sim \text{Normal}(u, \sigma^2)$, $\bar{x}_n \sim N(u, \frac{\sigma^2}{n})$

$\therefore x_i - \bar{x}_n \sim N(0, \frac{n+1}{n}\sigma^2) \rightarrow x_i, \bar{x}_n$ related. var 不能直接相加.

② $\therefore M_{x_i-\bar{x}_n}(t) = e^{\frac{(n+1)}{2n}\cdot\sigma^2\cdot t^2}$

$x_i - \frac{1}{n}\sum_{j=1}^n x_j = \frac{nx_i - (x_1+x_2+\dots+x_n)}{n} = \frac{(n-1)x_i}{n} - \frac{1}{n}\sum_{j\neq i} x_j$

x_i, x_j indep.

二维 MGF $M_{X,Y}(s,t) = E(e^{sX+tY})$

③ $M_{\bar{x}_n, x_i-\bar{x}_n}(s,t) = E(e^{s\cdot\bar{x}_n + t(x_i-\bar{x}_n)})$

$= E(e^{(s-t)\bar{x}_n + tx_i})$

$= E(e^{(s-t)\frac{1}{n}\sum_{j=1}^n x_j + tx_i})$

$= E(e^{(s-t)\frac{1}{n}x_i + (s-t)\frac{1}{n}\sum_{j\neq i}x_j + tx_i})$

$= E(e^{[\frac{1}{n}(s-t)+t]x_i}\prod_{j\neq i} e^{\frac{1}{n}(s-t)x_j})$ x_i indep

$= E(e^{[\frac{1}{n}(s-t)+t]x_i}]\cdot\prod_{j\neq i} E(e^{\frac{1}{n}(s-t)x_j})$ MGF of x_j

$= e^{u(\frac{1}{n}(s-t)+t)+\sigma^2[\frac{1}{n}(s-t)+t]^2/2}$.

$\prod_{j\neq i} e^{u[\frac{1}{n}(s-t)]+\sigma^2\cdot(\frac{1}{n}(s-t))^2/2}$ => n-1 个重复的

$= e^{u(\frac{1}{n}(s-t)+t)+\sigma^2[\frac{1}{n}(s-t)+t]^2/2}$
$e^{[u[\frac{1}{n}(s-t)]+\sigma^2\cdot(\frac{1}{n}(s-t))^2/2]\cdot n-1}$

$= \frac{u(\frac{1}{n}(s-t)+t)}{n} + \frac{\sigma^2}{2}\left(\frac{(s-t)^2}{n^2}+t^2+\frac{2}{n}(s-t)\cdot t\right)$

$+ \frac{u(\frac{1}{n}(s-t)+t)}{n}\cdot n-1 + \frac{\sigma^2(n-1)}{2n^2}(s-t)^2$

$\Rightarrow \frac{u(\frac{1}{n}(s-t)+t)}{\cancel{n}}(\cancel{1+n-1}) + \frac{\sigma^2(s-t)^2}{2n^{\cancel{2}}}(\cancel{1+n-1}) + \frac{\sigma^2}{2}(t^2+\frac{2}{n}(s-t)t)$

$\frac{u(s-t)}{n^2} + \frac{t}{n} + \frac{\sigma^2(s-t)^2}{2n} + \frac{\sigma^2t^2}{2} + \frac{2\sigma^2(s-t)t}{2n}$

$\frac{us-ut}{n^2} + \frac{t^2}{n^2} + \frac{\sigma^2(s-t)^2}{2n} + \frac{n\sigma^2t^2}{2n} + \frac{\sigma^2(s-t)t}{2n}$

$\frac{n\sigma^2t^2}{2n} + \frac{\sigma^2st-\sigma^2t^2}{2n}$

$\frac{us-ut}{n^2} + \frac{t^2}{n^2} + \frac{\sigma^2(s^2+t^2-2st)}{2n} + \frac{(n-1)\sigma^2t^2}{2n} + \frac{\sigma^2st}{2n}$

$\frac{us}{n^2} + \frac{t^2}{n^2} + \frac{\sigma^2s^2}{2n} + \frac{(n-1)\sigma^2t^2}{2n}$

$+ \frac{\sigma^2t^2}{2n} - \frac{\sigma^2st}{2n}$

① × ② $= e^{us+\frac{\sigma^2s^2}{n2}}\cdot e^{\frac{n-1}{2n}\sigma^2\cdot t^2}$

$= e^{us+\frac{\sigma^2s^2}{2n}+\frac{(n-1)\sigma^2t^2}{2n}}$

$x_i \sim N(u, \sigma^2)$,

$\frac{n-1}{n}x_i \sim N(\frac{n-1}{n}u, (\frac{n-1}{n})^2\sigma^2)$ A

$\sum_{j\neq i}x_j \sim N((n-1)u, (n-1)\sigma^2)$

$\frac{1}{n}\sum_{j\neq i}x_j \sim N(\frac{n-1}{n}u, \frac{n-1}{n^2}\sigma^2)$ B

$\Rightarrow A-B \sim N(\frac{n-1}{n}u - \frac{n-1}{n}u, \frac{(n-1)^2+n-1}{n^2}\sigma^2)$

$\sim N(0, \frac{n^2+1-2n+n-1}{n^2}\sigma^2)$

$\sim N(0, \frac{n^2-n}{n^2}\sigma^2)$

$\sim N(0, \frac{n-1}{n}\sigma^2)$

$\therefore M_{x_i-\bar{x}_n}(t) = e^{\frac{n-1}{n}\sigma^2\cdot t^2/2}$

$\frac{2us}{n^2} + \frac{t^2}{n^2} + \frac{2\sigma^2s^2}{2n^2} - \frac{n\sigma^2st}{2n^2}$

$\begin{cases} x_1 = ny_1 - y_2 - \bar{x} - y_3 - \bar{x} \dots - y_n - \bar{x} \\ x_1 = ny_1 - y_2 - y_3 \dots - y_n - n\bar{x} \\ x_2 = y_2 + \bar{x} \\ x_n = y_n + \bar{x} \end{cases}$

$\frac{dx_1}{dy_1} = n$ $\frac{dx_1}{dy_2} = -1$ .... $\frac{dx_1}{dy_n} = -1$

$\frac{dx_2}{dy_1} = 0$ $\frac{dx_2}{dy_2} = 1$ $\frac{dx_2}{dy_3} = 0$ $\frac{dx_2}{dy_n} = 0$

$\frac{dx_n}{dy_1} = 0$ . . . $\frac{dx_n}{dy_n} = 1$

$$\begin{vmatrix} n & -1 & -1 & -1 & \cdots -1 \\ 0 & 1 & 0 & 0 & \cdots 0 \\ 0 & 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 0 & 1 & \cdots 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = n$$

2017-(3d)

$S^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2 \Rightarrow (n-1)S^2 = \Sigma(x_i-\bar{x})^2 = \Sigma x_i^2 - n\bar{x}^2$

$(n-1)S^2 = \Sigma(x_i-\bar{x})^2$

Let $V^T = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots \frac{1}{\sqrt{n}})$ $V^TV = 1$.

Q be an orthogonal matrix, first row V^T. Y

$Y = Q\cdot X \Rightarrow \begin{cases} Y^TY = (QX)^TQX = X^TQ^TQX = X^TX. \\ Y_1 = (\frac{1}{\sqrt{n}}, \dots \frac{1}{\sqrt{n}})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \Sigma\frac{1}{\sqrt{n}}x_i = \frac{1}{\sqrt{n}}\Sigma x_i = \frac{n\bar{x}}{\sqrt{n}} = \sqrt{n}\bar{x} \\ \vdots \end{cases}$

$\Rightarrow \Sigma y_i^2 = \Sigma x_i^2$

$\therefore (n-1)S^2 = \Sigma x_i^2 - n\bar{x}^2$

$= \sum_{i=1}^n y_i^2 - y_1^2$

$= \underbrace{\sum_{i=1}^n x_i^2}_{\chi^2(p)} - y_1^2$

$Y_1 = \sqrt{n}\bar{x} \sim N(0, 1)$

$y_1^2 \sim \chi^2(1)$

$\chi^2(p) - \chi^2(1)$

$\sim \chi^2(p-1)$